Ground States for Elliptic Equations in $\mathbb{R}^2$ with Exponential Critical Growth

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Abstract

In this paper, we obtain a mountain pass characterization of ground state solutions for some class of elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical (exponential) growth range.

1 Introduction

This paper is concerned with the existence of solutions of a nonlinear scalar field equation of the form

$$-\Delta u = g(u) \quad \text{in } \mathbb{R}^2\,, \qquad u \in H^1(\mathbb{R}^2)\,, \tag{1}$$

and in particular we will study the following problem

$$-\Delta u + u = f(u) \quad \text{in } \mathbb{R}^2\,, \qquad u \in H^1(\mathbb{R}^2)\,, \tag{2}$$

that is, problem (1) with $g(s) := f(s) - s$.

The study of these kind of problems is motivated by applications in many areas of mathematical physics. In particular, solutions of (2) provide stationary states for the nonlinear Klein-Gordon equation and for the nonlinear Schrödinger equation.

Problem (1) has been extensively studied starting from the fundamental papers due to Berestycki and Lions [4] and to Berestycki, Gallouët and Kavian [5]. We recall that these papers are both concerned with *subcritical* nonlinearities, in particular in [4] the authors treated nonlinearities with subcritical *polynomial* growth, while in [5] the authors treated nonlinearities with subcritical *exponential* growth. From now on, we will focus our attention on the case when the nonlinear term is of exponential type, since our aim is to study problem (2) with a nonlinearity f exhibiting a critical exponential growth.

The maximal growth which can be treated variationally in the Sobolev space $H^1(\mathbb{R}^2)$ is given by the *Trudinger-Moser inequality*:

Theorem 1.1 ([10], Theorem 1.1). *There exists a constant $C > 0$ such that*

$$\sup_{u\in H^1(\mathbb{R}^2),\, \|u\|_{H^1}\le 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)\, dx \le C \tag{3}$$

where $\|u\|_{H^1}^2 := \|\nabla u\|_2^2 + \|u\|_2^2$ is the standard Sobolev norm. This inequality is sharp: if we replace the exponent 4π with any $\alpha > 4\pi$ the supremum is infinite.

In view of this inequality we say that a nonlinearity f has *critical growth* if there exists $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0 & \text{for } \alpha > \alpha_0 \ , \\ +\infty & \text{for } \alpha < \alpha_0 \ . \end{cases}$$

Our aim is to obtain a mountain pass characterization of ground state solutions of problem (2). The natural functional corresponding to a variational approach to problem (2) is

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + u^2)\, dx - \int_{\mathbb{R}^2} F(u)\, dx = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \int_{\mathbb{R}^2} G(u)\, dx\,, \quad u \in H^1(\mathbb{R}^2)\,,$$

where $F(s) := \int_0^s f(t)\, dt$ and $G(s) := \int_0^s g(t)\, dt$. We will say that I has a *mountain pass geometry*, if the following conditions hold:

(I_0) $I(0) = 0$;

(I_1) there exist ϱ, $a > 0$ such that $I(u) \ge a > 0$ for any $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{H^1} = \varrho$;

(I_2) there exists $u_0 \in H^1(\mathbb{R}^2)$ such that $\|u_0\|_{H^1} > \varrho$ and $I(u_0) < 0$.

We will always denote by $c \in \mathbb{R}$ the *mountain pass value*

$$c := \inf_{\gamma\in\Gamma} \sup_{t\in[0,\,1]} I(\gamma(t))\,, \qquad \Gamma := \left\{\gamma \in \mathcal{C}([0,\,1],\, H^1(\mathbb{R}^2)) \mid \gamma(0) = 0,\, I(\gamma(1)) < 0\right\} .$$

We recall also that a solution u of problem (2) is a *ground state* if $I(u) = m$ with

$$m := \inf\left\{I(u) \mid u \in H^1(\mathbb{R}^2) \setminus \{0\} \text{ is a solution of (2)}\right\} .$$

In [8], Jeanjean and Tanaka obtain a mountain pass characterization of ground state solutions for the more general nonlinear scalar field equation (1) in the case when the nonlinearity g (not necessarily of the form $f(s) - s$) has a *subcritical* exponential growth.

Theorem 1.2 ([8]). *Assume*

(g_0) $g : \mathbb{R} \to \mathbb{R}$ *is continuous and odd;*

(g_1) $\lim_{s\to 0} \frac{g(s)}{s} = -\nu < 0$ *;*

(g_2) *for any* $\alpha > 0$ *there exists* $C_\alpha > 0$ *such that* $|g(s)| \le C_\alpha e^{\alpha s^2}$ *for all* $s \ge 0$*;*

(g_3) *there exists* $s_0 > 0$ *such that* $G(s_0) > 0$.

Then the functional $I(u) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx - \int_{\mathbb{R}^2} G(u)\, dx$ *belongs to* $\mathcal{C}^1(H^1(\mathbb{R}^2),\, \mathbb{R})$ *and has a mountain pass geometry. Moreover the mountain pass value* c *is a critical value and* $0 < c = m$.

Recently Alves, Montenegro and Souto [3] improved the arguments in [8], assuming $g(s) = f(s) - s$ and considering nonlinearities with *critical* exponential growth.

Theorem 1.3 ([3]). *Assume that*

(f_0) $f : \mathbb{R} \to \mathbb{R}$ *is continuous and has critical exponential growth with* $\alpha_0 = 4\pi$ *;*

(f_1) $\lim_{s\to 0}\dfrac{f(s)}{s}=0$ *;*

(f_2) $0<2F(s)\le f(s)s$ *for any* $s\in\mathbb{R}\setminus\{0\}$ *,*

(f_η) *there exists* $\eta>0$ *and* $q\in(2,+\infty)$ *such that* $f(s)\ge \eta s^{q-1}$ *for all* $s\ge 0$.

If (f_η) *holds with*

$$\eta>\left(\frac{q-2}{q}\right)^{\frac{q-2}{q}} C_q^{\frac{q}{2}} \tag{4}$$

where $C_q>0$ *is the best constant of the Sobolev embedding* $H^1(\mathbb{R}^2)\hookrightarrow L^q(\mathbb{R}^2)$*, namely*

$$C_q\|u\|_q^2\le\|u\|_{H_1}^2 \qquad \forall u\in H^1(\mathbb{R}^2)\ .$$

Then the mountain pass value c is a critical value and gives the ground state level, namely $0<c=m$.

To obtain our results, we will follow some ideas introduced in [3].

2 Main results

Our main result is concerned with the particular case when $f(s)=\lambda s e^{4\pi s^2}$ where $0<\lambda<1$.

Theorem 2.1. *Let* $0<\lambda<1$ *and let*

$$f(s):=\lambda s e^{4\pi s^2} \qquad \forall s\in\mathbb{R}\ . \tag{5}$$

Then $I\in\mathcal{C}^1(H^1(\mathbb{R}^2),\mathbb{R})$ *has a mountain pass geometry, the mountain pass value c is a critical value and gives the ground state level, namely* $0<c=m$.

Moreover, replacing assumption (f_η) of Theorem 1.3 with the following more natural assumption

(f_3) $\lim_{|s|\to+\infty}\dfrac{sf(s)}{e^{4\pi s^2}}\ge\beta_0>0$,

we obtain the same result as in [3] (see Theorem 1.3 above).

Theorem 2.2. *Assume* (f_0), (f_1), (f_2) *and* (f_3). *Then* $I\in\mathcal{C}^1(H^1(\mathbb{R}^2),\mathbb{R})$ *has a mountain pass geometry, the mountain pass value c is a critical value and* $0<c=m$.

Comparing Theorem 2.2 with the result obtained by Alves, Montenegro and Souto (see Theorem 1.3 above), we see that their hypothesis (f_η) about the behavior of f near zero is replaced by an assumption, i.e. (f_3), at infinity.

We recall that assumption (f_3) for bounded domains was introduced in [1] (see also [6]) to obtain an existence result for elliptic equations with nonlinearities in the critical exponential growth range in bounded domains of $\mathbb{R}^2$. In a subsequent paper, [7], (f_3) was taken into account to prove an existence result for analogous equations in the whole space $\mathbb{R}^2$.

To prove Theorem 2.1 and Theorem 2.2 we will follow the methods of [3] which improve the ideas introduced in [8]. In the proof of Theorem 1.2 a key argument is the existence of a solution of problem (1) given in [5]. In [5] it was shown that under the assumptions (g_0), (g_1), (g_2) and (g_3)

the nonlinear scalar field equation (1) possesses a nontrivial ground state solution by means of the constrained minimization method

$$\inf \left\{ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \mid u \in H^1(\mathbb{R}^2) \setminus \{0\}, \int_{\mathbb{R}^2} G(u) \, dx = 0 \right\} .$$

The main difficulty, as highlighted in [3], for the proof of Theorem 1.3 is indeed to show that the infimum

$$A := \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \, dx \mid u \in H^1(\mathbb{R}^2) \setminus \{0\}, \int_{\mathbb{R}^2} G(u) \, dx = 0 \right\}$$

is achieved, provided that (f_0), (f_1), (f_2) and (f_η) with $\eta > 0$ as in (4) hold. Therefore we point out that, following [3], as a by-product of the proofs of Theorem 2.1 and Theorem 2.2 we have

Proposition 2.1. *Assume either f is of the form (5) with $0 < \lambda < 1$ or assume (f_0), (f_1), (f_2) and (f_3). Then A is attained and the minimizer is, under a suitable change of scale, a solution of problem (2). In particular $m \leq A$.*

This paper is organized as follows. In Section 3 we show that the functional I has a mountain pass geometry and in Section 4 we introduce some preliminary results. In Section 5 we obtain a precise estimate for the mountain pass level c that will enable us to prove, in Section 6, Proposition 2.1. Finally in Section 7 we prove the main theorems, Theorem 2.1 and Theorem 2.2, and the following

Proposition 2.2. *Assume either f is of the form (5) with $0 < \lambda < 1$ or assume (f_0), (f_1), (f_2) and (f_3). Then the minimizer $u \in H^1(\mathbb{R}^2)$ of A is a ground state solution of problem (2), that is $m = A$.*

3 Mountain pass geometry

If f is as in (5) with $0 < \lambda < 1$ then, for fixed $q > 2$ we have the existence of two constants c_1, $c_2 > 0$ such that

$$|f(s)| \leq c_1 |s| + c_2 |s|^{q-1} (e^{4\pi s^2} - 1) \qquad \forall s \in \mathbb{R} , \tag{6}$$

moreover, fixed $q > 2$ we have that for any $\varepsilon > 0$ there exists a constant $C(q, \varepsilon) > 0$ such that

$$F(s) \leq \left(\frac{\lambda}{2} + \varepsilon \right) s^2 + C(q, \varepsilon) |s|^q (e^{4\pi s^2} - 1) \qquad \forall s \in \mathbb{R} . \tag{7}$$

Note that (7) implies that $F(u) \in L^1(\mathbb{R}^2)$ for any $u \in H^1(\mathbb{R}^2)$ and thus the functional $I : H^1(\mathbb{R}^2) \to \mathbb{R}$ is well defined. Furthermore, from (6) and using standard arguments (see [4], Theorem A.VI), it follows that $I \in \mathcal{C}^1(H^1(\mathbb{R}^2), \mathbb{R})$.

Similarly in the case when (f_0) and (f_1) hold, fixed $q > 2$, for any $\alpha > 4\pi$ and any $\varepsilon > 0$ we have the existence of a constant $C(q, \alpha, \varepsilon) > 0$ such that

$$|f(s)| \leq \varepsilon |s| + C(q, \alpha, \varepsilon) |s|^{q-1} (e^{\alpha s^2} - 1) \qquad \forall s \in \mathbb{R} ,$$

and if in addition (f_2) holds then

$$F(s) \leq \frac{\varepsilon}{2} s^2 + C(q, \alpha, \varepsilon) |s|^q (e^{\alpha s^2} - 1) \qquad \forall s \in \mathbb{R} . \tag{8}$$

Therefore also in the case when (f_0), (f_1) and (f_2) hold we have that the functional I is well defined and of class $\mathcal{C}^1$ on $H^1(\mathbb{R}^2)$.

Obviously $I(0) = 0$, namely (I_0) holds. Now we prove that I satisfies also (I_1).

Lemma 3.1. *Assume either f is of the form (5) with $0 < \lambda < 1$ or assume (f_0), (f_1) and (f_2). Then there exist ϱ, $a > 0$ such that $I(u) \geq a > 0$ for any $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{H^1} = \varrho$.*

Proof. We begin considering the case when f is of the form (5) with $0 < \lambda < 1$. Fixed $q > 2$, for any $u \in H^1(\mathbb{R}^2)$ we have

$$\int_{\mathbb{R}^2} |u|^q (e^{4\pi u^2} - 1)\, dx \leq \|u\|_{2q}^q \left(\int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)^2\, dx \right)^{\frac{1}{2}} \leq \overline{C}_1 \|u\|_{H^1}^q \left(\int_{\mathbb{R}^2} (e^{8\pi u^2} - 1)\, dx \right)^{\frac{1}{2}}$$

where $\overline{C}_1 > 0$ is a constant independent of u and we used the fact that the embedding $H^1(\mathbb{R}^2) \hookrightarrow L^{2q}(\mathbb{R}^2)$ is continuous. Moreover, recalling the *Trudinger-Moser inequality* (3), we have the existence of a constant $\overline{C}_2 > 0$ such that

$$\int_{\mathbb{R}^2} (e^{8\pi u^2} - 1)\, dx = \int_{\mathbb{R}^2} \left(e^{8\pi \|u\|_{H^1}^2 \left(\frac{u}{\|u\|_{H^1}} \right)^2} - 1 \right) dx \leq \overline{C}_2$$

for any $u \in H^1(\mathbb{R}^2)$ with $8\pi \|u\|_{H^1}^2 \leq 4\pi$. Therefore applying (7), we get for any $\varepsilon > 0$

$$\int_{\mathbb{R}^2} F(u)\, dx \leq \left(\frac{\lambda}{2} + \varepsilon \right) \|u\|_{H^1}^2 + \overline{C}(q, \varepsilon) \|u\|_{H^1}^q \qquad \forall u \in H^1(\mathbb{R}^2),\ \|u\|_{H^1} \leq \frac{1}{\sqrt{2}}\ .$$

Let $0 < \varrho < \frac{1}{\sqrt{2}}$. Fixed $q > 2$, for any $\varepsilon > 0$

$$I(u) \geq \frac{1}{2}(1 - \lambda - 2\varepsilon)\varrho^2 - \overline{C}(q, \varepsilon)\varrho^q \qquad \forall u \in H^1(\mathbb{R}^2),\ \|u\|_{H^1} = \varrho\ ,$$

and choosing $\varepsilon > 0$ so that $1 - \lambda - 2\varepsilon > 0$ and ϱ sufficiently small we have that

$$I(u) \geq a := \frac{1}{2}(1 - \lambda - 2\varepsilon)\varrho^2 - \overline{C}(q, \varepsilon)\varrho^q > 0\ .$$

Using (8) and arguing as before, it easy to prove that I satisfies (I_1) also in the case when (f_0), (f_1) and (f_2) hold. □

We end this section with the proof of (I_2).

Lemma 3.2. *Assume either f is of the form (5) with $0 < \lambda < 1$ or assume (f_0) and (f_2). Then there exists $u_0 \in H^1(\mathbb{R}^2)$ such that $\|u_0\|_{H^1} > \varrho$ and $I(u_0) < 0$.*

Proof. We begin with the case when f is of the form (5). We fix $u \in H^1(\mathbb{R}^2)$. Using the definition of F and the power series expansion of the exponential function, we get

$$I(tu) \leq \frac{1}{2} t^2 \|u\|_{H^1} - \frac{\lambda}{2} t^2 \|u\|_2^2 - \lambda \pi t^4 \|u\|_4^4 \quad \forall t \geq 0\ ,$$

from which we deduce that $I(tu) \to -\infty$ as $t \to +\infty$. In the case when (f_0) and (f_2) hold, in particular for any $M > 0$ there exists $C_M > 0$ such that

$$F(s) \geq M|s|^2 - C_M \qquad \forall s \in \mathbb{R}\ .$$

Therefore, fixed $u \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, for any $t \geq 0$ we can estimate

$$I(tu) \leq \frac{1}{2} t^2 \|u\|_{H^1} - M t^2 \|u\|_2^2 + C_M |\mathrm{supp}\ u|$$

and choosing M sufficiently large we can conclude that $I(tu) \to -\infty$ as $t \to +\infty$. □

4 Preliminary results

Let $H^1_{\mathrm{rad}}(\mathbb{R}^2)$ be the space of spherically symmetric functions belonging to $H^1(\mathbb{R}^2)$,

$$H^1_{\mathrm{rad}}(\mathbb{R}^2) := \left\{ u \in H^1(\mathbb{R}^2) \mid u(x) = u(|x|) \text{ a.e. in } \mathbb{R}^2 \right\} .$$

Lemma 4.1. *Assume that f is of the form (5) with $0 < \lambda < 1$. Let $\{u_n\}_n \subset H^1_{rad}(\mathbb{R}^2)$ be a sequence satisfying*

$$\sup_n \|\nabla u_n\|_2^2 = \varrho < 1 \qquad \text{and} \qquad \sup_n \|u_n\|_2^2 = M < +\infty \, . \tag{9}$$

Then $u_n \rightharpoonup u \in H^1_{rad}(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^2} F(u_n)\, dx - \frac{\lambda}{2}\|u_n\|_2^2 \quad \overset{n\to+\infty}{\longrightarrow} \quad \int_{\mathbb{R}^2} F(u)\, dx - \frac{\lambda}{2}\|u\|_2^2 \, .$$

Before proceeding with the proof of this Lemma, we point out that the *Trudinger-Moser inequality* (3) holds also if we replace the standard Sobolev norm with the modified norm

$$\|u\|^2_{H^1,\,\tau} := \|\nabla u\|_2^2 + \tau \|u\|_2^2 \quad \forall u \in H^1(\mathbb{R}^2)$$

where $\tau > 0$. In fact in the proof of (3) given in [10] (see also [2]) the value $\tau = 1$, appearing in $\|\cdot\|_{H^1} = \|\cdot\|_{H^1,\,1}$ as a multiplicative constant for the L^2-norm, does not play any role and can be replaced by any $\tau > 0$. Therefore in [10] the author proved indeed that for any fixed $\tau > 0$

$$\sup_{u \in H^1(\mathbb{R}^2),\, \|u\|_{H^1,\,\tau} \leq 1} \int_{\mathbb{R}^2} (e^{4\pi u^2} - 1)\, dx < +\infty \, . \tag{10}$$

This will enable us to prove Lemma 4.1.

Proof of Lemma 4.1. Let $\{u_n\}_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^2)$ be a sequence satisfying (9) and $u_n \rightharpoonup u \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ in $H^1(\mathbb{R}^2)$. We have to show that

$$\int_{\mathbb{R}^2} P(u_n)\, dx \quad \overset{n\to+\infty}{\longrightarrow} \quad \int_{\mathbb{R}^2} P(u)\, dx,$$

where

$$P(s) := F(s) - \frac{\lambda}{2} s^2 = \lambda \Big[\frac{1}{8\pi}(e^{4\pi s^2} - 1) - \frac{1}{2} s^2 \Big].$$

To this aim, the idea is to apply the *compactness lemma of Strauss* (see Theorem A.I in [4]).

First, we notice that there exists $\alpha_0 > 4\pi$ such that

$$\sup_n \int_{\mathbb{R}^2} (e^{\alpha_0 u_n^2} - 1)\, dx < +\infty\ . \tag{11}$$

In fact, since $\varrho < 1$, there exists $\sigma > 0$ such that $\varrho < 1 - \sigma < 1$. Choosing $0 < \tau < \frac{1-(\sigma+\varrho)}{M}$, we have that $\|u_n\|^2_{H^1,\tau} < 1 - \sigma$ for any $n \geq 1$. Therefore applying (10), we can conclude that

$$\sup_n \int_{\mathbb{R}^2} (e^{\alpha u_n^2} - 1)\, dx < +\infty \qquad \text{for any } 0 < \alpha \leq \frac{4\pi}{1-\sigma}$$

and, in particular, this last inequality holds for $4\pi < \alpha \leq \frac{4\pi}{1-\sigma}$.

It is easy to see that

$$\lim_{s\to 0} \frac{P(s)}{e^{\alpha_0 s^2} - 1} = 0$$

and, since $\alpha_0 > 4\pi$, we have also that

$$\lim_{|s|\to+\infty} \frac{P(s)}{e^{\alpha_0 s^2} - 1} = 0\ .$$

Moreover, recalling that the embedding $H^1_{\mathrm{rad}}(\mathbb{R}^2) \hookrightarrow L^p(\mathbb{R}^2)$ is compact for any $p \in (2, +\infty)$, we have that $u_n \to u$ a.e. in $\mathbb{R}^2$ and this together with the continuity assumption on f leads us to deduce that $P(u_n) \to P(u)$ a.e. in $\mathbb{R}^2$. Finally, we can notice that $u_n(x) \to 0$ as $|x| \to +\infty$ uniformly with respect to n, as a consequence of the following *radial lemma*:

$$|v(x)| \leq \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{|x|}} \|v\|_{H^1} \qquad \text{a.e. in } \mathbb{R}^2, \tag{12}$$

which holds for any $v \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$.

Then, applying the *compactness lemma of Strauss*, we can conclude that $P(u_n)$ converges to $P(u)$ in $L^1(\mathbb{R}^2)$ as $n \to +\infty$. □

We recall that in [3] the authors proved the following result

Lemma 4.2. *Assume* (f_0) *and* (f_1)*. Let* $\{u_n\}_n \subset H^1_{rad}(\mathbb{R}^2)$ *be a sequence satisfying conditions (9) of Lemma 4.1. Then* $u_n \rightharpoonup u \in H^1_{rad}(\mathbb{R}^2)$ *in* $H^1(\mathbb{R}^2)$ *and*

$$\int_{\mathbb{R}^2} F(u_n)\, dx \to \int_{\mathbb{R}^2} F(u)\, dx\ .$$

We can notice that the proof of this lemma can be achieved arguing as in the proof of Lemma 4.1 but letting $P(s) := F(s)$.

We now prove that the infimum A is strictly positive, but before we point out that whenever we deal with a minimizing sequence for A, that is a sequence $\{u_n\}_n \subset H^1(\mathbb{R}^2) \setminus \{0\}$ such that

$$\int_{\mathbb{R}^2} G(u_n)\, dx = 0 \quad \forall n \geq 1 \qquad \text{and} \qquad \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\, dx \quad \overset{n\to+\infty}{\longrightarrow} \quad A\ ;$$

without loss of generality we may assume that $\{u_n\}_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$ and that $\|u_n\|_2 = 1$. In fact if $\{u_n\}_n \subset H^1(\mathbb{R}^2) \setminus \{0\}$ is a minimizing sequence for A then the sequence $\{u_n^*\}_n \subset H^1(\mathbb{R}^2) \setminus \{0\}$, where u_n^* is the spherically symmetric decreasing rearrangement of u_n, is a minimizing sequence too. Furthermore letting

$$v_n(x) := u_n(x\|u\|_2) \qquad \text{for a.e. } x \in \mathbb{R}^2$$

for any $n \geq 1$, we have that

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla v_n|^2 = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 \,, \quad \int_{\mathbb{R}^2} G(v_n)\,dx = \frac{1}{\|u_n\|_2^2}\int_{\mathbb{R}^2} G(u_n)\,dx = 0$$

and $\|v_n\|_2 = 1$.

Lemma 4.3. *Assume either f is of the form (5) with $0 < \lambda < 1$, or assume (f_0) and (f_1). Then $A > 0$.*

Proof. In the case that we assume (f_0) and (f_1), since Lemma 4.2 holds, we can argue as in the proof of [3], Lemma 5.3 to conclude that $A > 0$. Therefore we only consider the case when $f(s) := \lambda s e^{4\pi s^2}$ with $0 < \lambda < 1$. Obviously $A \geq 0$ and we argue by contradiction assuming that $A = 0$. Then there exists $\{u_n\}_n \subset H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$ with $\|u_n\|_2 = 1$ $\forall n \geq 1$ and

$$\int_{\mathbb{R}^2} G(u_n)\,dx = 0 \quad \forall n \geq 1 \,, \qquad \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\,dx \quad \overset{n\to+\infty}{\longrightarrow} \quad 0 \,.$$

Let $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ be the weak limit of $\{u_n\}_n$ in $H^1(\mathbb{R}^2)$, then from Lemma 4.1 it follows that

$$\int_{\mathbb{R}^2} F(u_n)\,dx - \frac{\lambda}{2} = \int_{\mathbb{R}^2} F(u_n)\,dx - \frac{\lambda}{2}\|u_n\|_2^2 \quad \overset{n\to+\infty}{\longrightarrow} \quad \int_{\mathbb{R}^2} F(u)\,dx - \frac{\lambda}{2}\|u\|_2^2 \,.$$

Since

$$0 = \int_{\mathbb{R}^2} G(u_n)\,dx = \int_{\mathbb{R}^2} F(u_n)\,dx - \frac{1}{2}\|u_n\|_2^2 = \int_{\mathbb{R}^2} F(u_n)\,dx - \frac{1}{2} \,, \qquad \text{i.e.} \quad \int_{\mathbb{R}^2} F(u_n)\,dx = \frac{1}{2} \,, \tag{13}$$

we have that

$$\int_{\mathbb{R}^2} F(u)\,dx - \frac{\lambda}{2}\|u\|_2^2 = \frac{1}{2}(1-\lambda) > 0$$

from which it follows that $u \neq 0$. On the other hand, the weak convergence $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ implies that

$$0 = \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\,dx \geq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \geq 0 \,,$$

that is $u \equiv 0$, which leads to a contradiction. $\square$

We introduce the set $\mathcal{P}$ of non-trivial functions satisfying the Pohozaev identity

$$\mathcal{P} := \left\{ u \in H^1(\mathbb{R}^2) \setminus \{0\} \;\middle|\; \int_{\mathbb{R}^2} G(u)\,dx = 0 \right\}$$

and we can notice that

$$A = \inf_{u\in\mathcal{P}} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\,dx \,.$$

Since $A > 0$, arguing as in the proof of [8], Lemma 4.1 we have the following result

Lemma 4.4. *Assume either f is of the form (5) with $0 < \lambda < 1$, or assume (f_0) and (f_1). Then*

$$\gamma([0,\, 1]) \cap \mathcal{P} \neq 0 \qquad \gamma \in \Gamma\ .$$

This lemma leads to the following relation between the infimum A and the mountain pass level c

Lemma 4.5. *Assume either f is of the form (5) with $0 < \lambda < 1$, or assume (f_0) and (f_1). Then the infimum A satisfies the inequality $A \leq c$.*

Proof. Let $\gamma \in \Gamma$ and let $t_0 \in (0,\, 1]$ be such that $\gamma(t_0) \in \mathcal{P}$, the existence of such a t_0 is guaranteed by Lemma 4.4. Since $\gamma(t_0) \in \mathcal{P}$, we have

$$I(\gamma(t_0)) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx$$

and thus

$$\max_{t\in[0,\, 1]} I(\gamma(t)) \geq I(\gamma(t_0)) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2\, dx \geq A\ . \tag{14}$$

From the arbitrary choice of $\gamma \in \Gamma$, inequality (14) holds for any $\gamma \in \Gamma$ and hence $c \geq A$. $\square$

5 Estimate of the mountain pass level c

In order to get an upper bound for the mountain pass level c we will show the existence of $u \in H^1(\mathbb{R}^2)$ such that

$$\max_{t\geq 0} I(tu) < \frac{1}{2}\ . \tag{15}$$

This gives indeed more precise information about the mountain pass level c, in fact from (15) it is easy to deduce the existence of $\gamma \in \Gamma$ such that

$$c \leq \max_{t\in[0,\, 1]} I(\gamma(t)) < \frac{1}{2}\ . \tag{16}$$

First, we consider the case when f is as in (5) with $0 < \lambda < 1$. To obtain the existence of $u \in H^1(\mathbb{R}^2)$ which satisfies the inequality (15), the fact that

$$\lim_{|s|\to+\infty} \frac{sf(s)}{e^{4\pi s^2}} = +\infty \tag{17}$$

plays an important role. In particular we can notice from (17) it follows that for fixed

$$\beta_0 > \frac{1}{\pi} \tag{18}$$

there exists $\overline{s} = \overline{s}(\beta_0) > 0$ such that

$$sf(s) \geq \beta_0 e^{4\pi s^2} \quad \forall |s| \geq \overline{s}\ . \tag{19}$$

We consider the modified *Moser sequence* introduced in [6]:

$$\overline{\omega}_n(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{\frac{1}{2}} & 0 \le |x| \le \frac{1}{n} \ , \\ \frac{\log \frac{1}{|x|}}{(\log n)^{\frac{1}{2}}} & \frac{1}{n} \le |x| \le 1 \ , \\ 0 & |x| \ge 1 \ . \end{cases}$$

We have $\overline{\omega}_n \in H_0^1(B_1) \subset H^1(\mathbb{R}^2)$, $\|\nabla \overline{\omega}_n\|_2 = 1$ and $\|\overline{\omega}_n\|_2^2 = \mathcal{O}(1/\log n)$. Then we define

$$\omega_n := \frac{\overline{\omega}_n}{\|\overline{\omega}_n\|_{H^1}} \ .$$

Lemma 5.1. *Assume f is of the form (5) with $0 < \lambda < 1$. Then there exists $n \ge 1$ such that*

$$\max_{t \ge 0} I(t\omega_n) < \frac{1}{2} \ .$$

Proof. We argue by contradiction assuming that for any $n \ge 1$ we have $\max_{t\ge 0} I(t\omega_n) \ge \frac{1}{2}$. For any $n \ge 1$, let $t_n > 0$ be such that $I(t_n\omega_n) = \max_{t \ge 0} I(t\omega_n) \ge 1/2$, then we can estimate

$$\frac{1}{2} \le I(t_n\omega_n) = \frac{1}{2} t_n^2 \|\omega_n\|_{H^1}^2 - \int_{\mathbb{R}^2} F(t_n\omega_n)\, dx \le \frac{1}{2} t_n^2$$

and thus $t_n^2 \ge 1$, $\forall n \ge 1$. At $t = t_n$ we have

$$0 = \frac{d}{dt} I(t\omega_n)\Big|_{t=t_n} = t_n - \int_{\mathbb{R}^2} f(t_n\omega_n)\omega_n\, dx \ ,$$

which implies that

$$t_n^2 = \int_{\mathbb{R}^2} f(t_n\omega_n) t_n \omega_n\, dx \ . \tag{20}$$

We claim that $\{t_n\}_n \subset \mathbb{R}$ is bounded. In fact, since

$$t_n\omega_n = \frac{t_n}{\|\overline{\omega}_n\|_{H^1}} \frac{1}{\sqrt{2\pi}} \sqrt{\log n} \to +\infty \qquad \text{in } B_{\frac{1}{n}} \ ,$$

it follows from (19) that at least for $n \ge 1$ sufficiently large

$$t_n^2 \ge \int_{B_{\frac{1}{n}}} f(t_n\omega_n) t_n\omega_n\, dx \ge \beta_0 \int_{B_{\frac{1}{n}}} e^{4\pi (t_n\omega_n)^2}\, dx = \frac{\pi}{n^2} \beta_0 e^{2 \frac{t_n^2}{\|\overline{\omega}_n\|_{H^1}^2} \log n} \ . \tag{21}$$

Consequently

$$1 \ge \pi \beta_0 e^{2 \frac{t_n^2}{\|\overline{\omega}_n\|_{H^1}^2} \log n - 2\log t_n - 2\log n}$$

for $n \ge 1$ sufficiently large, and thus $\{t_n\}_n$ must be bounded.

We claim that $t_n^2 \to 1$ as $n \to +\infty$. Since $t_n^2 \ge 1$ $\forall n \ge 1$, we argue by contradiction assuming that $\lim_{n\to+\infty} t_n^2 > 1$. Recalling (21), for $n \ge 1$ sufficiently large we have

$$t_n^2 \ge \pi \beta_0 e^{2 \log n \left(\frac{t_n^2}{\|\overline{\omega}_n\|_{H^1}^2} - 1 \right)}$$

and letting $n \to +\infty$ we get a contradiction with the boundedness of the sequence $\{t_n\}_n$.

In order to estimate (20) more precisely, we define the sets $A_n := \{x \in B_1 \mid t_n\omega_n(x) \geq \overline{s}\}$ and $C_n := B_1 \setminus A_n$ where $\overline{s} > 0$ is given in (19). With (20) and (19) we can estimate for any $n \geq 1$

$$t_n^2 \geq \int_{B_1} f(t_n\omega_n)t_n\omega_n\,dx \geq \beta_0 \int_{B_1} e^{4\pi t_n^2\omega_n^2}\,dx + \int_{C_n} f(t_n\omega_n)t_n\omega_n\,dx - \beta_0 \int_{C_n} e^{4\pi t_n^2\omega_n^2}\,dx \tag{22}$$

Since $\omega_n \to 0$ a.e. in B_1, from the definition of C_n we obtain that the characteristic functions $\chi_{C_n} \to 1$ a.e. in B_1, and the Lebesgue dominated convergence theorem implies that

$$\int_{C_n} f(t_n\omega_n)t_n\omega_n\,dx \to 0\,, \qquad \int_{C_n} e^{4\pi t_n^2\omega_n^2}\,dx \to \pi \qquad \text{as } n \to +\infty\,.$$

If we prove that

$$\lim_{n\to+\infty} \int_{B_1} e^{4\pi t_n^2\omega_n^2}\,dx \geq 2\pi \tag{23}$$

then by (22) $1 = \lim_{n\to+\infty} t_n^2 \geq \pi\beta_0$ which is in contradiction with (18). To end the proof it remains only to show that inequality (23) holds. Since $t_n^2 \geq 1$, we have

$$\int_{B_1} e^{4\pi t_n^2\omega_n^2}\,dx \geq \int_{B_1\setminus B_{\frac{1}{n}}} e^{4\pi\omega_n^2}\,dx = 2\pi \int_{\frac{1}{n}}^{1} e^{\frac{2}{\|\overline{\omega}_n\|_{H^1}^2}\frac{1}{\log n}\log^2\left(\frac{1}{s}\right)} s\,ds$$

and if we make the change of variable

$$\tau = \frac{\log\frac{1}{s}}{\|\overline{\omega}_n\|_{H^1}\log n}$$

then we obtain the following estimate

$$\int_{B_1\setminus B_{\frac{1}{n}}} e^{4\pi t_n^2\omega_n^2}\,dx \geq 2\pi\|\overline{\omega}_n\|_{H^1}\log n \int_0^{\frac{1}{\|\overline{\omega}_n\|_{H^1}}} e^{2\log n(\tau^2 - \,\tau\|\overline{\omega}_n\|_{H^1})}\,d\tau\,.$$

Now it suffices to notice that

$$\tau^2 - \tau\|\overline{\omega}_n\|_{H^1} \geq \begin{cases} -\|\tau\overline{\omega}_n\|_{H^1} & ,\ 0 \leq \tau \leq \frac{1}{2\|\overline{\omega}_n\|_{H^1}} \\ \left(\frac{2}{\|\overline{\omega}_n\|_{H^1}} - \|\overline{\omega}_n\|_{H^1}\right)\left(\tau - \frac{1}{\|\overline{\omega}_n\|_{H^1}}\right) + \frac{1}{\|\overline{\omega}_n\|_{H^1}^2} - 1 & ,\ \frac{1}{2\|\overline{\omega}_n\|_{H^1}} \leq \tau \leq \frac{1}{\|\overline{\omega}_n\|_{H^1}} \end{cases}$$

to conclude that (23) holds. $\square$

Next, we consider the case when (f_2) and (f_3) hold. In this case, as a consequence of (f_3), we have that for any $\varepsilon > 0$ there exists $s_\varepsilon > 0$ such that $sf(s) \geq (\beta_0 - \varepsilon)e^{4\pi s^2}$ $\forall |s| \geq s_\varepsilon$. Let $r > 0$ be such that $\beta_0 > 1/(r^2\pi)$, we consider the modified *Moser's sequence* introduced in [7]:

$$\overline{M}_n(x) := \frac{1}{\sqrt{2\pi}} \begin{cases} (\log n)^{\frac{1}{2}} & 0 \leq |x| \leq \frac{r}{n}\,, \\ \frac{\log\frac{r}{|x|}}{(\log n)^{\frac{1}{2}}} & \frac{r}{n} \leq |x| \leq r\,, \\ 0 & |x| \geq r\,. \end{cases}$$

Arguing as before (see also [7], Lemma 4.4) we have the following result

Lemma 5.2. *Assume* (f_2) *and* (f_3). *Then there exists* $n \in \mathbb{N}$ *such that*

$$\max_{t\geq 0} I(tM_n) < \frac{1}{2} \qquad \textit{where } M_n := \frac{\overline{M}_n}{\|\overline{M}_n\|_{H^1}} \,.$$

6 The infimum A is attained

In this Section we will prove Proposition 2.1. We can notice that in either case, when f is of the form (5) with $0 < \lambda < 1$, or when (f_0), (f_1), (f_2) and (f_3) hold, if the infimum A is attained then the minimizer $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$ is a solution of problem (2), under a suitable change of scale. In fact, if $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$ is such that

$$\frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \, dx = A \qquad \text{and} \qquad \int_{\mathbb{R}^2} G(u)\, dx = 0$$

then there exists a Lagrange multiplier $\theta \in \mathbb{R}$, namely

$$\int_{\mathbb{R}^2} \nabla u \cdot \nabla v \, dx = \theta \int_{\mathbb{R}^2} g(u) v \, dx \qquad \forall v \in H^1(\mathbb{R}^2) \,.$$

Claim: The Lagrange multiplier θ is positive.

Proof of the claim. First, we can notice that the case $\theta = 0$ does not occur, since by assumption $u \neq 0$. We infer that $\theta > 0$. In fact, suppose by contradiction that $\theta < 0$. Then arguing as in [4], we can find $w \in H^1(\mathbb{R}^2)$ satisfying

$$\int_{\mathbb{R}^2} G(u + \varepsilon w)\, dx > 0 \qquad \text{and} \qquad \|\nabla(u + \varepsilon w)\|_2^2 < \|\nabla u\|_2^2$$

for some $\varepsilon > 0$ sufficiently small. Moreover, we may assume that $u + \varepsilon w \neq 0$. Now we define $h \in \mathcal{C}([0,\, 1],\, \mathbb{R})$ as

$$h(t) := \int_{\mathbb{R}^2} G(t[u + \varepsilon w])\, dx.$$

By construction $h(0) = 0$ and $h(1) > 0$.

Assume that f is of the form (5) with $0 < \lambda < 1$. Then from (7) it follows that

$$\begin{aligned}\Big| \int_{\mathbb{R}^2} F(t[u+\varepsilon w])\, dx \Big| &\leq \Big(\frac{\lambda}{2} + \varepsilon\Big) t^2 \|u + \varepsilon w\|_2^2 + C(q,\, \varepsilon) t^q \int_{\mathbb{R}^2} |u + \varepsilon w|^q (e^{4\pi t^2 (u+\varepsilon w)^2} - 1)\, dx \leq \\ &\leq \Big(\frac{\lambda}{2} + \varepsilon\Big) t^2 \|u + \varepsilon w\|_2^2 + C(q,\, \varepsilon) t^q \|u + \varepsilon w\|_{2q}^q \Big(\int_{\mathbb{R}^2} (e^{8\pi t^2 (u+\varepsilon w)^2} - 1)\, dx\Big)^{\frac{1}{2}}.\end{aligned}$$

Since $0 \leq t^2 \|u + \varepsilon w\|_{H^1}^2 \to 0$ as $t \to 0$, there exists $t_0 \in (0,\, 1)$ such that for $0 < t < t_0$ we have

$$\Big| \int_{\mathbb{R}^2} F(t[u+\varepsilon w])\, dx \Big| \leq \Big(\frac{\lambda}{2} + \varepsilon\Big) t^2 \|u + \varepsilon w\|_2^2 + \tilde{C}(q,\, \varepsilon) t^q \|u + \varepsilon w\|_{2q}^q,$$

as a consequence of the *Trudinger-Moser inequality* (3). Thus, for $0 < t < t_0$

$$h(t) \leq \Big(\frac{\lambda}{2} + \varepsilon - \frac{1}{2}\Big) t^2 \|u + \varepsilon w\|_2^2 + \tilde{C}(q,\, \varepsilon) t^q \|u + \varepsilon w\|_{2q}^q$$

and choosing $\varepsilon > 0$ so small that $\frac{\lambda}{2} + \varepsilon - \frac{1}{2} < 0$, we deduce that that $h(t) < 0$ for sufficiently small $t > 0$. In the case when (f_0), (f_1), (f_2) and (f_3) hold, we can achieve the same conclusion applying (8).

Hence, in either case, when f is of the form (5) or when (f_0), (f_1), (f_2) and (f_3) hold, we have $h(t) < 0$ for sufficiently small $t > 0$. Consequently, there exists $t_1 \in (0, 1)$ such that $h(t_1) = 0$, which means that $t_1(u + \varepsilon w) \in H^1(\mathbb{R}^2)$ satisfies the constraint condition

$$\int_{\mathbb{R}^2} G(t_1[u + \varepsilon w])\, dx = 0$$

and, since u is a minimizer for A,

$$\frac{1}{2}\|\nabla u\|_2^2 \le \frac{1}{2}\|t_1 \nabla(u + \varepsilon w)\|_2^2 < \frac{1}{2}\|\nabla(u + \varepsilon w)\|_2^2 < \frac{1}{2}\|\nabla u\|_2^2 \,.$$

This is a contradiction and θ must be positive; hence the claim is proved.

Since $\theta > 0$, we can set

$$u_\theta(x) := u\left(\frac{x}{\sqrt{\theta}}\right) \qquad \text{for a.e. } x \in \mathbb{R}^2 \ . \tag{24}$$

Then u_θ is a non-trivial solution of problem (2) and hence $m \le I(u_\theta)$. Moreover

$$\int_{\mathbb{R}^2} |\nabla u_\theta|^2\, dx = \int_{\mathbb{R}^2} |\nabla u|^2\, dx = A \ , \quad \int_{\mathbb{R}^2} G(u_\theta)\, dx = \theta \int_{\mathbb{R}^2} G(u)\, dx = 0 \ ,$$

from which we get $I(u_\theta) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_\theta|^2\, dx = A$ and thus $m \le A$.

Therefore to prove Proposition 2.1, it remains to show that the infimum A is achieved. The proof in the case in which we assume (f_0), (f_1), (f_2) and (f_3) can be easily reduced to the proof of [3], Theorem 1.4. It suffices to notice that from Lemma 4.5 and from inequality (16) it follows that $A < 1/2$, and thus we are in the same framework of the proof of [3], Theorem 1.4.

Proof of Proposition 2.1 in the case $f(s) := \lambda s e^{4\pi s^2}$ $\forall s \in \mathbb{R}$ with $0 < \lambda < 1$. From Lemma 4.5 and from inequality (16), it follows that $A < 1/2$.

Let $\{u_n\}_n \in H^1_{\mathrm{rad}}(\mathbb{R}^2) \setminus \{0\}$, $\|u_n\|_2 = 1$, $\forall n \ge 1$, be a minimizing sequence for A:

$$\int_{\mathbb{R}^2} G(u_n)\, dx = 0 \qquad \forall n \ge 1 \qquad \text{and} \qquad \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2\, dx \quad \overset{n \to +\infty}{\longrightarrow} \quad A \ . \tag{25}$$

We will prove that the weak limit $u \in H^1_{\mathrm{rad}}(\mathbb{R}^2)$ of $\{u_n\}_n$ in $H^1(\mathbb{R}^2)$ is a minimizer for A.

Since

$$\limsup_{n \to +\infty} \int_{\mathbb{R}^2} |\nabla u_n|^2\, dx = 2A < 1 \ ,$$

the assumptions of Lemma 4.1 are satisfied. Arguing as in (13), we deduce that

$$\int_{\mathbb{R}^2} F(u)\, dx - \frac{\lambda}{2}\|u\|_2^2 = \frac{1}{2}(1 - \lambda) > 0 \tag{26}$$

which in particular implies that $u \ne 0$.

From the weak convergence $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, we get

$$A = \liminf_{n\to+\infty} \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_n|^2 \geq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 \, .$$

Let

$$h(t) := \int_{\mathbb{R}^2} G(tu)\,dx = \int_{\mathbb{R}^2} F(tu)\,dx - \frac{t^2}{2}\|u\|_2^2 \quad \forall t > 0 \, ;$$

to conclude the proof it suffices to prove that $h(1) = 0$. Since $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$, we have $\|u\|_2^2 \leq \liminf_{n\to+\infty} \|u_n\|_2^2 = 1$ and this together with (26) gives

$$\begin{aligned} h(1) &= \int_{\mathbb{R}^2} F(u)\,dx - \frac{1}{2}\|u\|_2^2 = \int_{\mathbb{R}^2} F(u)\,dx - \frac{\lambda}{2}\|u\|_2^2 + \frac{1}{2}(\lambda-1)\|u\|_2^2 = \\ &= \frac{1}{2}(1-\lambda) + \frac{1}{2}(\lambda-1)\|u\|_2^2 = \frac{1}{2}(1-\lambda)(1-\|u\|_2^2) \geq 0 \, . \end{aligned}$$

We argue by contradiction assuming that $h(1) \neq 0$, that is $h(1) > 0$. Using the definition of F and the power series expansion of the exponential function, for any $t \in (0,\,1)$ we have

$$\int_{\mathbb{R}^2} F(tu)\,dx \quad \leq \quad \frac{\lambda}{2}t^2\|u\|_2^2 + t^4\frac{\lambda}{8\pi}\sum_{j=2}^{+\infty}\frac{(4\pi)^j}{j!}\int_{\mathbb{R}^2} u^{2j}\,dx \leq \frac{\lambda}{2}t^2\|u\|_2^2 + t^4\frac{\lambda}{8\pi}\int_{\mathbb{R}^2}(e^{4\pi u^2}-1)\,dx \, .$$

Hence for any $t \in (0,\,1)$

$$h(t) \leq \frac{1}{2}(\lambda-1)t^2\|u\|_2^2 + t^4\frac{\lambda}{8\pi}\int_{\mathbb{R}^2}(e^{4\pi u^2}-1)\,dx \, ,$$

from which we deduce that $h(t) < 0$ for $t > 0$ sufficiently small. But, by assumption, $h(1) > 0$ and thus there exists $t_0 \in (0,\,1)$ such that $h(t_0) = 0$. Consequently, recalling the definition of h, we have

$$A \leq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla(t_0 u)|^2\,dx = \frac{1}{2}t_0^2\int_{\mathbb{R}^2}|\nabla u|^2\,dx \leq t_0^2 A < A$$

which is a contradiction. □

7 Proofs of Theorem 2.1 and Theorem 2.2

In order to prove Theorem 2.1 and Theorem 2.2 we can notice that, both in the case when f is of the form (5) with $0 < \lambda < 1$ and in the case when (f_0), (f_1), (f_2) and (f_3) hold, from Proposition 2.1 we have $m \leq A$. Moreover, Lemma 4.5 tells us that $A \leq c$ and hence $m \leq c$. It remains only to show that

$$m \geq c \tag{27}$$

to conclude that the mountain pass level c gives the ground state level.

In [8] the authors proved the following result

Theorem 7.1 ([8], Lemma 2.1)**.** *Assume (g_0), (g_1), (g_2) and (g_3) as in Theorem 1.2. Then for any solution u of (1) there exists a path $\gamma \in \Gamma$ such that $u \in \gamma([0,\,1])$ and*

$$\max_{t\in[0,\,1]} I(\gamma(t)) = I(u) \, .$$

It is easy to see that the proof of this theorem works also under our assumptions and this leads to (27).

Indeed, we can notice that in this way we proved that $m = A = c$. Hence if $u \in H^1(\mathbb{R}^2)$ is a minimizer for A and we define u_θ as in (24) then u_θ is a ground state solution of problem (2). This gives the proof of Proposition 2.2.

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